Dynamically ordered energy function for Morse-Smale diffeomorphisms on 3-manifolds

Viatcheslav Grines, Francois Laudenbach, Olga Pochinka

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V. Grines* F. Laudenbach† O. Pochinka‡

Abstract

This note deals with arbitrary Morse-Smale diffeomorphisms in dimension 3 and extends ideas from [3], [4], where gradient-like case was considered. We introduce a kind of Morse-Lyapunov function, called dynamically ordered, which fits well dynamics of diffeomorphism. The paper is devoted to finding conditions to the existence of such an energy function, that is, a function whose set of critical points coincides with the non-wandering set of the considered diffeomorphism. We show that the necessary and sufficient conditions to the existence of a dynamically ordered energy function reduces to the type of embedding of one-dimensional attractors and repellers, each of them is a union of zero- and one-dimensional unstable (stable) manifolds of periodic orbits of a given Morse-Smale diffeomorphism on a closed 3-manifold.

1 Introduction and formulation of the results

Let M be a closed jrientable 3-manifold and $f : M \to M$ be a preserving orientation Morse-Smale diffeomorphism, that is: its nonwandering set Ω_f is finite, hence consists of periodic points; f is hyperbolic along Ω_f and the stable and unstable manifolds have transverse intersections.

Definition 1 *A Morse function $\varphi : M \to \mathbb{R}$ is said to be a Lyapunov function for f if:*

1) $\varphi(f(x)) < \varphi(x)$ for every $x \notin \Omega_f$;

2) $\varphi(f(x)) = \varphi(x)$ for every $x \in \Omega_f$.

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Sometimes we shall speak of a Lyapunov function even when it is only defined on some domain $N \subset M$, meaning that the above conditions 1), 2) hold only for points $x \in N$ such that $f(x) \in N$.

Let us recall that a C^2-smooth function $\varphi : M \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate. Using ideas from [12] it is possible to construct Lyapunov functions for f. For this aim, one considers the suspension of f, a 4-dimensional manifold which is fibered over the circle and is endowed with a Morse-Smale vector field X transverse to the fibration. The method introduced by S. Smale in [12] for constructing Lyapunov function for Morse-Smale vector fields without closed orbits can be extended to suspension and allows one to construct a Lyapunov function Φ for X. The restriction of Φ to the base fibre, identified with M, is a Lyapunov function for diffeomorphism f.

According to statement 5 below the periodic points of f are critical points of its Lyapunov function φ and the index of φ at $p \in \Omega_f$ equals the dimension of W^u_p. At the same time any periodic point p is a maximum of the restriction of φ to the unstable manifold W^u_p and a minimum of its restriction to the stable manifold W^s_p. If these extrema are non-degenerate then the invariant manifolds of p are transversal to all regular level sets of φ in some neighborhood of the point p. This local property is useful for the construction of a (global) Lyapunov function. Next definition was introduced in [4].

Definition 2 *A Lyapunov function $\varphi : M \to \mathbb{R}$ for the Morse-Smale diffeomorphism $f : M \to M$ is called a Morse-Lyapunov function if every periodic point p is a non-degenerate maximum (resp. minimum) of the restriction of φ to the unstable (resp. stable) manifold W^u_p (resp. W^s_p).*

Among the Lyapunov functions of f those which are Morse-Lyapunov form a generic set in the C^∞-topology (see [4], theorem 1). In general, a Morse-Lyapunov function may have critical points which are not periodic points of f.

Definition 3 *A Morse-Lyapunov function φ is called an energy function for a Morse-Smale diffeomorphism f if the set of critical points of φ coincides with Ω_f.*

D. Pixton in [9] established an existence of energy function for any Morse-Smale diffeomorphisms given on closed smooth two-dimensional manifold and constructed a gradient-like diffeomorphism on $\mathbb{S}^3$ which has no energy function. According to S. Smale [12] any Morse-Smale flow without closed trajectories (gradient-like flow) given on closed smooth manifold of any dimension possesses by an energy function. Thus there is an actual problem a finding of conditions to an existence of energy function for Morse-Smale diffeomorphisms. First step in this direction was made by the authors for gradient-like diffeomorphisms in the papers [3], [4].

Let us recall that a Morse-Smale diffeomorphism $f : M \to M$ is called *gradient-like* if for any pair of periodic points x, y ($x \neq y$) the condition $W^u_x \cap W^s_y \neq \emptyset$ implies $\dim W^s_x < \dim W^s_y$. It follows from the definition that a Morse-Smale diffeomorphism is gradient-like if and only if there are no *heteroclinic points* that is, intersection points of two-dimensional and one-dimensional invariant manifolds of different saddle points. Notice that two-dimensional invariant manifolds of different saddle points of a gradient-like diffeomorphism may have a non-empty intersection along the so-called *heteroclinic curves* (see figure 1).

In [3], [4] (Theorem 4) we gave necessary and sufficient conditions to the existence of a *self-indexing* energy function for a Morse-Smale diffeomorphism $f : M \to M$ and showed that a non gradient-like diffeomorphisms do not possess a self-indexing energy function. Here self-indexing means $\varphi(p) = \dim W^u_p$ for every point $p \in \Omega_f$.

In the present paper we introduce the notion of *dynamically ordered* Morse-Lyapunov function for an arbitrary Morse-Smale diffeomorphism on 3-manifold. By using the above-mentioned arguments, such a function will exist easily if it is not required to be an energy function. We will show that the existence of such an energy function depends on how the one-dimensional attractors (and repellers) embed into the ambient manifold. More details are given below.

Let $f : M \to M$ be a Morse-Smale diffeomorphism. Following to S. Smale we introduce a partial order $\prec$ on the set of periodic orbits of f in the following way:

$$\mathcal{O}_p \prec \mathcal{O}_r \iff W^s_{\mathcal{O}_p} \cap W^u_{\mathcal{O}_r} \neq \emptyset\,.$$

This definition means intuitively that all wandering points flow down along unstable manifolds to smaller elements. A sequence of different periodic orbits $\mathcal{O}_p = \mathcal{O}_{p_0}, \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_k} = \mathcal{O}_r$ ($k \geq 1$) such that $\mathcal{O}_{p_0} \prec \mathcal{O}_{p_1} \prec \ldots \prec \mathcal{O}_{p_k}$ is called a *chain of length k connecting $\mathcal{O}_r$ to $\mathcal{O}_p$*. The maximum length of such chains is called, by J. Palis in [8], the *behaviour* of $\mathcal{O}_r$ relative to $\mathcal{O}_p$ and is denoted by $beh(\mathcal{O}_r|\mathcal{O}_p)$. For completeness it is assumed $beh(\mathcal{O}_r|\mathcal{O}_p) = 0$ if $W^u_{\mathcal{O}_r} \cap W^s_{\mathcal{O}_p} = \emptyset$.

For each $q \in \{0, 1, 2, 3\}$, denote Ω_q the subset of periodic points r such that dim $W^u_r = q$ and denote k_q the number of periodic orbits in the set Ω_q. Set $k_f = k_0 + k_1 + k_2 + k_3$ the number of all periodic orbits. For each periodic orbit $\mathcal{O}_r$ we set $q_{\mathcal{O}_r} = \dim\ W^u_{\mathcal{O}_r}$ and $b_{\mathcal{O}_r} = \max\limits_{p\in\Omega_0}\{beh(\mathcal{O}_r|\mathcal{O}_p)\}$.

Definition 4 *A numbering of the periodic orbits: $\mathcal{O}_1, \ldots, \mathcal{O}_{k_f}$ is called dynamical if it satisfies to following conditions:*

1) if $q_{\mathcal{O}_i} < q_{\mathcal{O}_j}$ then $i < j$;

2) if $q_{\mathcal{O}_i} = q_{\mathcal{O}_j}$ and $b_{\mathcal{O}_i} < b_{\mathcal{O}_j}$ then $i < j$.

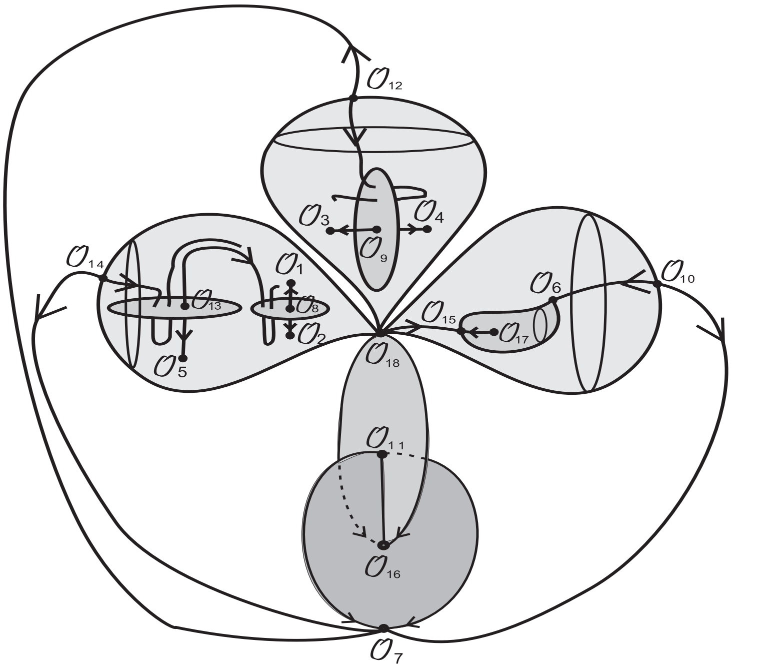

Figure 1: Phase portrait of a Morse-Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with dynamical numbering of the periodic orbits

Notice that any dynamical numbering preserves the partial order $\prec$ (that is,$\mathcal{O}_i \prec \mathcal{O}_j$ implies $i \leq j$). Indeed, as the intersection $W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j}$ is transverse, the condition $\mathcal{O}_i \prec \mathcal{O}_j$ implies the inequality dim $W^s_{\mathcal{O}_i}$ + dim $W^u_{\mathcal{O}_j} - 3 \geq 0$. Then $3 - q_{\mathcal{O}_i} + q_{\mathcal{O}_j} - 3 \geq 0$ and, hence, $q_{\mathcal{O}_i} \leq q_{\mathcal{O}_j}$. If $q_{\mathcal{O}_i} < q_{\mathcal{O}_j}$ then $i < j$ due to 1). If $q_{\mathcal{O}_i} = q_{\mathcal{O}_j}$ then the condition $\mathcal{O}_i \prec \mathcal{O}_j$ implies or $\mathcal{O}_i = \mathcal{O}_j$ and, hence, $i = j$, either $b_{\mathcal{O}_i} < b_{\mathcal{O}_j}$ and, hence, $i < j$ due to 2).

On figure 1 it is represented a phase portrait of a Morse-Smale diffeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ with Ω_f consisting of fixed points which are dynamically numerated.

Definition 5 *Let $\mathcal{O}_1, \ldots, \mathcal{O}_{k_f}$ be a dynamical numbering of the periodic orbits of f. A Morse-Lyapunov function φ for f is said to be dynamically ordered when $\varphi(\mathcal{O}_i) = i$ for $i \in \{1, \ldots, k_f\}$.*

For each $i = 1, \ldots, k_1$, set $A_i = \bigcup\limits_{j=1}^{i} W^u_{\mathcal{O}_j}$. It is known that the set A_i is an *attractor*, that is it has *a trapping neighborhood* M_i, which is a compact set such that $f(M_i) \subset int\ M_i$ (M_i is f-compressed) and $\bigcap\limits_{k \geq 0} f^k(M_i) = A_i$ (see, for example, [10]). Denote by r_i the number of saddles, by s_i the number of sinks and by c_i the number of connected components in A_i. Set $g_i = c_i + r_i - s_i$.

Let us recall that a smooth compact orientable three-dimensional manifold is called *a handlebody of a genus* $g \geq 0$ if it is diffeomorphic to a manifold which is obtained from a closed 3-ball by an orientation reversing identification of g pairs of pairwise disjoint closed 2-discs in its boundary. The boundary of such a handlebody is an orientable surface of genus g.

Definition 6 *A trapping neighborhood M_i of the attractor A_i is called a handle if: M_i consists of c_i handlebodies. The sum g_{M_i} of genera of all connected components of M_i is called genus of the handle neighborhood.*

Notice that for each $i = 1, \dots, k_0$, the number g_i equals 0, the attractor A_i is zero-dimensional (as it consists of the sink orbits) and has a handle neighborhood M_i of genus $g_i = 0$ consisting of c_i pairwise disjoint 3-balls (it follows, for example, from statement 6 below). For each $i = k_0 + 1, \dots, k_1$ the attractor A_i contains an one-dimensional connected component, therefor we will say (taking liberty) that A_i is one-dimensional attractor.

Proposition 1 *Each one-dimensional attractor A_i of Morse-Smale diffeomorphism $f : M \to M$ has a handle trapping neighborhood M_i with genus $g_{M_i} \geq g_i$.*

Definition 7 *A handle neighborhood M_i of one-dimensional attractor A_i is said to be tight if:*

1) $g_{M_i} = g_i$;

2) $W^s_\sigma \cap M_i$ consists of exactly one two-dimensional closed disc for each saddle point $\sigma \in \mathcal{O}_i$.

A one-dimensional attractor A_i possessing tight trapping neighborhood M_i is said to be tightly embedded.

By definition a *repeller* for f is an attractor for f^{-1}. Moreover, dynamical numbering of the orbits $\mathcal{O}_1, \dots, \mathcal{O}_{k_f}$ of a diffeomorphism f induces a dynamical numbering of the orbits $\tilde{\mathcal{O}}_1, \dots, \tilde{\mathcal{O}}_{k_f}$ of a diffeomorphism f^{-1} next way: $\tilde{\mathcal{O}}_i = \mathcal{O}_{k_f - i}$. Then a one-dimensional repeller for f is said to be *tightly embedded* if it is such an attractor for f^{-1} according to induced numbering.

Notice that the property for a one-dimensional attractor (repeller) to be tightly embedded gives a topological information about the embedding of the unstable manifolds of its saddle periodic points. In the example which was constructed by D. Pixton in [9] the unique one-dimensional attractor $A_3 = cl\, W^u_\sigma$ has the following property: $g_3 = 0$ but any 3-ball around $cl\, W^u_\sigma$ intersects W^s_σ at more than one 2-disc (see figure 2, where are drawn the phase portrait of Pixton's diffeomorpfism f and a 3-ball). Hence, this one-dimensional attractor is not tightly embedded.

Our main results are the following theorems.

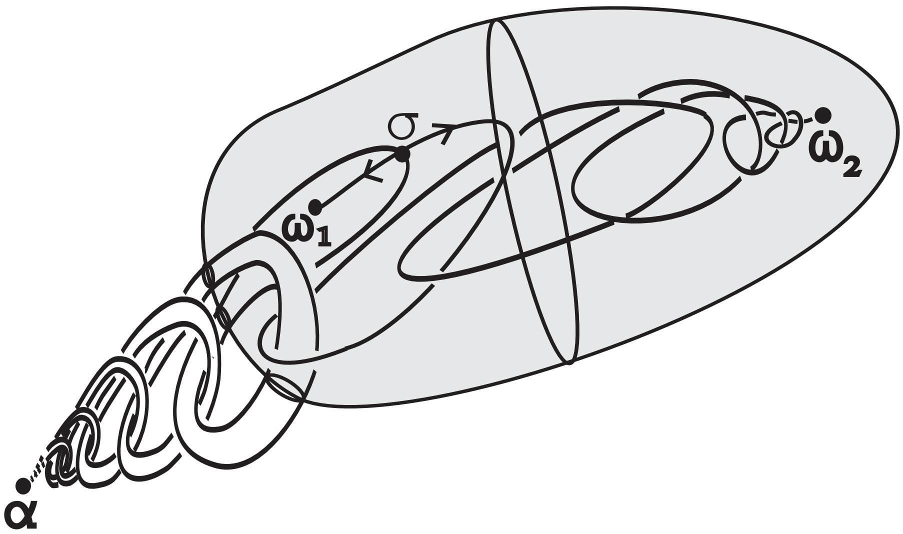

Figure 2: One-dimensional attractor of Pixton's example is not tightly embedded

Theorem 1 *If a Morse-Smale diffeomorphism $f : M \to M$ possesses a dynamically ordered energy function, then all one-dimensional attractors and repellers of f are tightly embedded.*

Definition 8 *A tight trapping neighborhood M_i of a one-dimensional attractor A_i is called strongly tight if $M_i \setminus A_i$ is diffeomorphic to $\partial M_i \times (0, 1]$. A one-dimensional attractor A_i possessing a strongly tight trapping neighborhood M_i is said to be strongly tightly embedded.*

Theorem 2 *Let f be a Morse-Smale diffeomorphism on a closed 3-manifold M. If all one-dimensional attractors and repellers of f are strongly tightly embedded, then f possesses a dynamically ordered energy function.*

Notice that the condition in the last theorem is not necessary. For example in section 5 of paper [4] there was constructed a diffeomorphism on $\mathbb{S}^2 \times \mathbb{S}^1$ possessing a dynamically ordered energy function, but whose one-dimensional attractor and repeller are not strongly tightly embedded.

The next theorem states a criterion for the existence of some dynamically ordered energy function for a Morse-Smale diffeomorphism without heteroclinic curves given on $\mathbb{S}^3$. Methods from [1] for realizing Morse-Smale diffeomorphisms show that this class is not empty. Moreover, it contains diffeomorphisms with chains of intersections of saddle invariant manifolds of arbitrary length (see figure 3, where it is represented a phase portrait of a diffeomorphism from the class under consideration). The criterion is based on paper [2], where it is specified interrelation between topology of the ambient 3-manifold

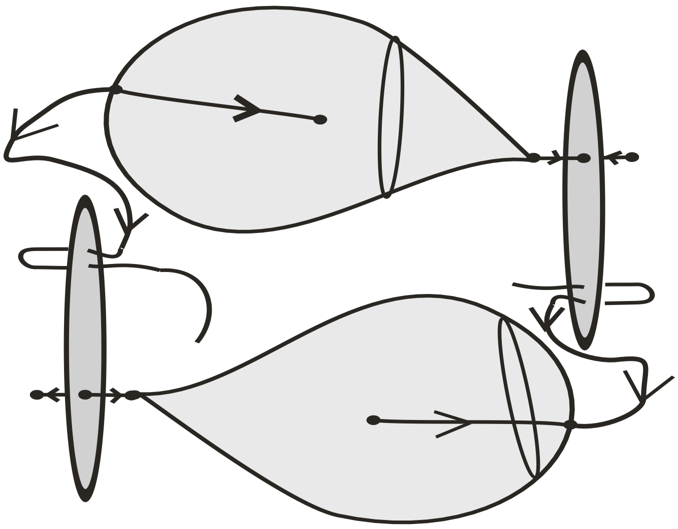

Figure 3: A Morse-Smale diffeomorphism without heteroclinic curves given on $\mathbb{S}^3$

M and structure of the non-wandering set of a Morse-Smale diffeomorphism without heteroclinic curves given on M. In particular, for any diffeomorphism of $\mathbb{S}^3$ without heteroclinic curves, the number r of all saddles and the number l of all sinks and sources satisfy the equality $r = l - 2$ (see statement 7 below). This equality implies that $g_i = 0$ for any one-dimensional attractor A_i. Thus, tightly embedded attractor A_i is strong tightly embedded. Applying results of theorems 1, 2 we get next criterion.

Theorem 3 *A Morse-Smale diffeomorphism* $f : \mathbb{S}^3 \to \mathbb{S}^3$ *without heteroclinic curves possesses a dynamically ordered energy function if and only if each one-dimensional attractor and repeller is tightly embedded.*

ACKNOWLEDGMENTS

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2 Auxiliary facts

In this section, we recall some statements that we need in the proof and give references.

Statement 1 *(λ-lemma, [8]). Let p be a hyperbolic fixed point of a diffeomorphism $f : M^n \to M^n$, $\dim W^u_p = \ell$, $0 < \ell < n$, Let $B^u \subset W^u_p$ and $B^s \subset W^s_p$ be small ℓ-disc and $(n-\ell)$-disc respectively centered at p. Let $V := B^u \times B^s$ be their product in a chart about p. Let B be an ℓ-disc transverse to W^s_p at x. Then, for any $\varepsilon > 0$, there exists a positive integer k_0 such that the connected component of $f^k(B) \cap V$ containing $f^k(x)$ is ε-C^1-close to B_u for each $k \geq k_0$.*

Definition 9 *Let F be a compact smooth surface properly embedded in a 3-manifold W (that is, $\partial F \subset \partial W$). Then F is called compressible in one from two following cases:*

1) there is a non contractible simple closed curve $c \subset int\ F$ and smoothly embedded 2-disk $D \subset int\ W$ such that $D \cap F = \partial D = c$;

2) there is a 3-ball $B \subset int\ W$ such that $F = \partial B$.

The surface F is said to be incompressible[1] *in W if it is not compressible in W.*

Statement 2 ([13], corollary 3.2) *Let S_g be an orientable surface of genus $g \geq 1$ and let F be an incompressible orientable surface properly embedded in $S_g \times [0,1]$ such that $\partial F \subset S_g \times \{1\}$. Then there is a surface $F_1 \subset S_g \times \{1\}$ which is homeomorphic to F, such that $\partial F = \partial F_1$ and $F \cup F_1$ bounds domain Δ in $S_g \times [0,1]$ such that $cl\Delta$ is homeomorphic to $F \times [0,1]$, where $cl(\cdot)$ stands for the closure.*

A particular case of statement 2 is the following fact.

Corollary 1 ([5], theorem 3.3) *Let S_g be a closed orientable surface of genus $g \geq 1$ and let surface $F \subset int(S_g \times [0,1])$ be a closed surface which has genus g and does not bound a domain in $S_g \times [0,1]$. Then F is incompressible in $S_g \times [0,1]$ and the closure of each connected component of $S_g \times [0,1] \setminus F$ is homeomorphic to $S_g \times [0,1]$.*

Proof: According to the preceding statement, it is sufficient to check that F is incompressible in $S_g \times [0,1]$. If F is compressible, there exists some incompressible surface F' whose genus g' is less than g and which still does not bound a domain in $S_g \times [0,1]$. So F' is not a sphere and $g' > 0$. As F' is incompressible, the preceding statement tells us that F' is diffeomorphic to S_g. Contradiction. $\diamond$

Statement 3 ([4], lemma 3.3) *For any Morse-Smale diffeomorphism $f : M \to M$ we have that $1 + |\Omega_1| - |\Omega_0| = 1 + |\Omega_2| - |\Omega_3|$, where $|\cdot|$ stands for the cardinality.*

[1]It is well known to topologists that a bicollared surface, different from sphere, is incompressible if and only if the inclusion $S \hookrightarrow W$ induces an injection of fundamental groups.

Statement 4 ([7], theorem 5.2) Let M^n be a closed manifold, $\varphi : M^n \to \mathbb{R}$ be a Morse function, C_q be the number of all its critical points with index q, $\beta_q(M^n)$ be the q-th Betti number and $\chi(M^n)$ be the Euler characteristic. Then $\beta_q(M^n) \leq C_q$ and $\chi(M^n) = \sum\limits_{q=0}^{n}(-1)^q C_q$.

Statement 5 *Let $\varphi : M^n \to \mathbb{R}$ be a Lyapunov function for a Morse-Smale diffeomorphism $f : M^n \to M^n$. Then*

1) $-\varphi$ is Lyapunov function for f^{-1};

2) if p is a periodic point of f then $\varphi(x) < \varphi(p)$ for every $x \in W^u_p \setminus p$ and $\varphi(x) > \varphi(p)$ for every $x \in W^s_p \setminus p$;

3) if p is a periodic point of f then p is a critical point of φ whose index is $\dim W^u_p$.

Statement 6 ([4], lemma 2.2) *Let $f : M^n \to M^n$ be a Morse-Smale diffeomorphism on an n-dimensional manifold and let $\mathcal{O}$ be a periodic orbit. For $p \in \mathcal{O}$, set $q = \dim W^u_p$. Then, there is some neighborhood U and an energy function $\varphi : U \to \mathbb{R}$ for f such that $(W^u_p \cap U) \subset Ox_1 \dots x_q$, $(W^s_p \cap U) \subset Ox_{q+1} \dots x_n$ for Morse coordinates $x_1, \dots, x_n$ of φ near p.*

Statement 7 ([2], theorem) *Let M be a three-dimensional closed, connected, orientable manifold. Let $f : M \to M$ be any Morse-Smale diffeomorphism without heteroclinic curves whose non-wandering set consists of r saddles and l nodes (sinks and sources). Then $m = \frac{r-l+2}{2}$ is non negative integer and following facts hold:*

1) if $m = 0$, then M is the 3-sphere;

2) if $m > 0$, then M is the connected sum of m copies $\mathbb{S}^2 \times \mathbb{S}^1$.

Conversely, for any non negative integers r, l, m such that $m = \frac{r-l+2}{2}$ is non negative integer, there exists 3-manifold M and some Morse-Smale diffeomorphism $f : M \to M$ with following properties:

a) M is 3-sphere if $m = 0$ and M is the connected sum of m copies $\mathbb{S}^2 \times \mathbb{S}^1$ if $m > 0$;

b) the non-wandering set of f consists of r saddles and l sinks and sources, the wandering set of f has no heteroclinic curves.

3 On one-dimensional attractors

Proof of proposition 1

For each $i = k_0+1, \dots, k_1$ let us prove the existence of handle neighborhood M_i for one-dimensional attractor A_i with $g_{M_i} \geq g_i$.

Proof: According to statement 6, there is a neighborhood $U_{A_{k_0}} \subset W^s_{A_{k_0}}$ of zero-dimensional attractor A_{k_0} and an energy function $\varphi_{A_{k_0}} : U_{A_{k_0}} \to \mathbb{R}$ for f such that $\varphi_{A_{k_0}}(A_{k_0}) = 0$ and for small $\varepsilon > 0$ each connected component of set

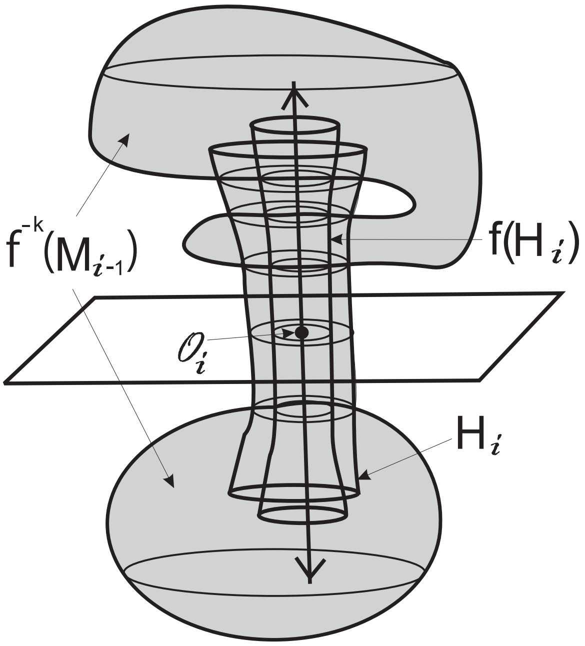

Figure 4: Construction of a handle neighborhood for one-dimensional attractor

$M_{k_0} = \varphi^{-1}_{A_{k_0}}((-\infty, \varepsilon])$ reads $\{(x_1, x_2, x_3) \in U_{A_{k_0}} \ : \ x_1^2 + x_2^2 + x_3^2 \leq \varepsilon\}$ in local coordinates x_1, x_2, x_3. Then M_{k_0} is trapping neighborhood of zero-dimensional attractor A_{k_0}, which is a union of c_{k_0} pairwise disjoint 3-balls. By induction on $i = k_0 + 1, \dots, k_1$ we construct a handle trapping neighborhood M_i for A_i.

Let $i = k_0 + 1$. Set $S_{k_0} = \partial M_{k_0}$. Without loss of generality we can suppose that S_{k_0} intersects $W^u_{\mathcal{O}_{k_0+1}}$ transversely; let n_{k_0} be the number of intersection points. Set $V_{k_0} = W^s_{\Omega_f \cap A_{k_0}} \setminus A_{k_0}$. Then the quotient $\hat{V}_{k_0} = V_{k_0}/f$ is made of the cobordism $M_{k_0} \setminus int\, f(M_{k_0})$ by gluing its boundaries by f. Hence, $\hat{V}_{k_0}$ is smooth orientable 3-manifold without boundary and natural projection $p_{_{k_0}} : V_{k_0} \to \hat{V}_{k_0}$ is cover. Then $p_{_{k_0}}(W^u_{\mathcal{O}_{k_0+1}})$ is a pair of knots which intersects $p_{_{k_0}}(S_{k_0})$ transversely at n_{k_0} points. Thus, there is a tubular neighborhood $\hat{T}^u_{k_0+1} \subset \hat{V}_{k_0}$ of $p_{_{k_0}}(W^u_{\mathcal{O}_{k_0+1}})$ such that $\hat{T}^u_{k_0+1} \cap p_{_{k_0}}(S_{k_0})$ consists of n_{k_0} 2-discs.

Set $T^u_{k_0+1} = p^{-1}_{k_0}(\hat{T}^u_{k_0+1})$. According to the λ-lemma (see statement 1), $T^u_{k_0+1} \cup W^s_{\mathcal{O}_{k_0+1}}$ is a neighborhood of $\mathcal{O}_{k_0+1}$. According to statement 6, there are some neighborhood $U_{\mathcal{O}_{k_0+1}} \subset (T^u_{k_0+1} \cup W^s_{\mathcal{O}_{k_0+1}})$ of $\mathcal{O}_{k_0+1}$ and an energy function $\varphi_{\mathcal{O}_{k_0+1}} : U_{\mathcal{O}_{k_0+1}} \to \mathbb{R}$ for f with $\varphi_{\mathcal{O}_{k_0+1}}(\mathcal{O}_{k_0+1}) = 0$. When $\varepsilon > 0$ is small enough, each connected component of $H_{k_0+1} = \varphi^{-1}_{\mathcal{O}_{k_0+1}}((-\infty, \varepsilon])$ reads $\{(x_1, x_2, x_3) \in U_{\mathcal{O}_{k_0+1}} \ : \ -x_1^2 + x_2^2 + x_3^2 \leq \varepsilon\}$ in local coordinates x_1, x_2, x_3. According to the λ-lemma, when $k \in \mathbb{N}$ is large enough, $f^{-k}(S_{k_0})$ intersects both H_{k_0+1} and $f(H_{k_0+1})$, its intersection with these domains consists of n_{k_0}

2-discs and $f(H_{k_0+1}) \setminus int\, f^{-k}(M_{k_0}) \subset int\, H_i$ (see figure 4). Thus, $M_{k_0+1} = f^{-k}(M_{k_0}) \cup H_{k_0+1}$ is a union of handlebodies, as it is obtained from the union of 3-balls $f^{-k}(M_{k_0})$ by gluing one-handles $H_{k_0+1} \setminus int\, f^{-k}(M_{k_0})$. Let us show that $f(M_{k_0+1}) \subset int\, M_{k_0+1}$.

Indeed, it is true for a point $x \in f^{-k}(M_{k_0})$ as $f^{-k}(M_{k_0})$ is f-compressed and it is true for a point $x \in (H_{k_0+1} \setminus f^{-k}(M_{k_0}))$ as $f(H_{k_0+1}) \setminus int\, f^{-k}(M_{k_0}) \subset int\, H_{k_0+1}$.

Let us prove the equality $\bigcap\limits_{k\geq 0} f^k(M_{k_0+1}) = A_{k_0+1}$. As $A_{k_0+1} \subset M_{k_0+1}$ and $f^k(A_{k_0+1}) = A_{k_0+1}$ for $k \in \mathbb{Z}$ then $A_{k_0+1} \subset \bigcap\limits_{k\geq 0} f^k(M_{k_0+1})$. Let us set $A'_{k_0+1} = \bigcap\limits_{k\geq 0} f^k(M_{k_0+1})$ and show that $A'_{k_0+1} = A_{k_0+1}$. Assume contrary: there is a point $x \in (A'_{k_0+1} \setminus A_{k_0+1})$. Due to theorem 2.3 in [11] there is a point $p \in (\Omega_f \setminus A_{k_0+1})$ such that $x \in W^u_p$. As the set A'_{k_0+1} is closed and invariant then $cl\ (\mathcal{O}_x) \subset A'_{k_0+1}$ and, hence, $p \in A'_{k_0+1}$. This is a contradiction with the fact $A'_{k_0+1} \subset W^s_{A_{k_0+1}\cap\Omega_f}$.

Recall that we denote by r_i the number of saddles, by s_i the number of sinks, by c_i the number of connected components of the attractor A_i and set $g_i = c_i + r_i - s_i$. By the construction M_{k_0+1} consists of c_{k_0+1} 3-balls with 1-handles[2]. Denote by $g_{M_{k_0+1}}$ the sum of genera of connected components of M_{k_0+1}. Let us show that $g_{M_{k_0+1}} \geq g_{k_0+1}$.

In denotation above, the number of points in the orbit $\mathcal{O}_{k_0+1}$ equals $r_{k_0+1} - r_{k_0}$. As $A_{k_0+1} = A_{k_0} \cup W^u_{\mathcal{O}_{k_0+1}}$ and $cl\ W^u_{\mathcal{O}_{k_0+1}} \setminus W^u_{\mathcal{O}_{k_0+1}} \subset A_{k_0}$ then $c_{k_0+1} \leq c_{k_0}$. Denote by l_{k_0+1} the number of connected components of the set $H_{k_0+1} \setminus int\, f^{-k}(M_{k_0})$. By the construction each of them is 1-handle and removing of $(l_{k_0+1} - (c_{k_0} - c_{k_0+1}))$ 1-handles from M_{k_0+1} gives the set with the same c_{k_0+1} connected components. Then the sum of genera $g_{M_{k_0}}$ of M_{k_0} can be calculate by formula $g_{M_{k_0}} = g_{M_{k_0+1}} - (l_{k_0+1} - (c_{k_0} - c_{k_0+1}))$. By the construction $l_{k_0+1} \geq (r_{k_0+1} - r_{k_0})$, hence $g_{M_{k_0}} \leq g_{M_{k_0+1}} - (r_{k_0+1} - r_{k_0} - (c_{k_0} - c_{k_0+1}))$ and $g_{M_{k_0+1}} \geq g_{M_{k_0}} + r_{k_0+1} - r_{k_0} - c_{k_0} + c_{k_0+1}$. As $g_{M_{k_0}} = g_{k_0}$ then $g_{M_{k_0+1}} \geq c_{k_0} + r_{k_0} - s_{k_0} + r_{k_0+1} - r_{k_0} - c_{k_0} + c_{k_0+1} = c_{k_0+1} + r_{k_0+1} - s_{k_0}$. As $s_{k_0} = s_{k_0+1}$ then $g_{M_{k_0+1}} \geq g_{k_0+1}$.

A smoothing of the set M_{k_0+1} is the required handle trapping neighborhood.

Assuming that handle neighborhood for attractor A_{i-1} already constructed, repeating construction above (changing k_0 by $i-1$), we construct f-compressed set $M_i = f^{-k}(M_{i-1}) \cup H_i$, being a union of handle neighborhood $f^{-k}(M_{i-1})$ with 1-handles $H_i \setminus int\, f^{-k}(M_{i-1})$. It is similar proved that M_i is required handle neighborhood. $\diamond$

[2]Recall that a 3-dimensional 1-handle is the product of an interval with a 2-disc.

Proposition 2 *The one-dimensional attractor A_{k_1} is connected.*

Proof: Firstly, let us prove that any trapping neighborhood M_{k_1} of A_{k_1} is connected. Let us assume the contrary: M_{k_1} is a union of pairwise disjoint closed sets B_1 and B_2. As M_{k_1} is f-compressed then without loss of generality we can suppose that $f(B_i) \subset int\, B_i,\ i = 1, 2$. By construction, $U_1 = \bigcup\limits_{k>0} f^{-k}(int\, B_1)$, $U_2 = \bigcup\limits_{k>0} f^{-k}(int\, B_2)$ are pairwise disjoint open sets and $U_1 \cup U_2 = W^s_{\Omega_0 \cup \Omega_1}$. On the other hand $W^s_{\Omega_0\cup\Omega_1} = M \setminus W^s_{\Omega_2\cup\Omega_3}$ and, hence, $W^s_{\Omega_0\cup\Omega_1}$ is connected as dim $M = 3$ and dim $W^s_{\Omega_2\cup\Omega_3} \leq 1$. This is a contradiction.

Thus A_i is connected as intersection of nested connected compact sets $M_i \supset f(M_i) \supset \ldots \supset f^k(M_i) \supset \ldots$. $\diamond$

4 Necessary condition for existence of dynamically ordered energy function

Proof of theorem 1

Let us prove that if a Morse-Smale diffeomorphism $f : M \to M$ has a dynamically ordered energy function, then its one-dimensional attractors and repellers are tightly embedded.

Proof: Notice that f and f^{-1} possess dynamically ordered energy functions simultaneously. Indeed, if $\varphi : M \to \mathbb{R}$ is such function for f then $-\varphi : M \to \mathbb{R}$ is an energy function for f^{-1} (see statement 5) and $\tilde{\varphi} = k_f + 1 - \varphi : M \to \mathbb{R}$ is dynamically ordered energy function for f^{-1}. Therefore, it is enough to prove the fact for attractors.

Let $\varphi : M \to \mathbb{R}$ be a dynamically ordered energy function for $f : M \to M$, $i = k_0+1, \ldots, k_1$ and $M_i = \varphi^{-1}([1, i+\varepsilon_i]), \varepsilon_i > 0$. It follows from properties of dynamically ordered energy function and statement 5 that any orbit $\mathcal{O}_j$ with number $j \leq i$ belongs to M_i. Due to statement 5, $W^u_{\mathcal{O}_j} \subset M_i$. Thus $A_i \subset M_i$. It follows from definition of Lyapunov function that $f(M_i) \subset int\ M_i$. Similar to proposition 1 it is proved equality $\bigcap\limits_{k\geq 0} f^k(M_i) = A_i$. Thus, M_i is trapping neighborhood of attractor A_i. Then M_i has the same number of connected components as A_i. Let us prove that there is $\varepsilon_i > 0$ such that M_i is a tight neighborhood of A_i.

As φ is a Morse-Lyapunov function then there is $\varepsilon_i > 0$ such that $W^s_\sigma \cap M_i$ consists of exactly one closed 2-disc for each saddle point $\sigma \in \mathcal{O}_i$. It follows from properties of dynamically ordered energy function and statement 5 that $\varphi|_{M_i}$ has exactly r_i+s_i critical points, among of them s_i points have index 0 and r_i points have index 1. According to Morse theory, M_i is a union of of s_i 3-balls with gluing of r_i 1-handles and hence is a union of c_i handlebodies. Denote by g_{M_i} the sum of genus of handlebodies from M_i. According to statement 4,

$\chi(M_i) = s_i - r_i$. It follows from Morse theory that M_i has the homotopy type of a cellular complex consisting of s_i zero-dimensional and r_i one-dimensional cells, then $-g_{_{M_i}} + c_i = m_i - r_i$ or $g_{_{M_i}} = g_i$. $\diamond$

5 Construction of a dynamically ordered energy function for f

Now f is a Morse-Smale diffeomorphism on a closed 3-manifold M and its one-dimensional attractors and repellers are strongly tightly embedded. Construction of a dynamically ordered energy function for f is based on technical lemmas of next section.

Recall that, by assumption of theorem 2, each one-dimensional attractor $A_i,\ i = k_0 + 1, \dots, k_1$ is strongly tightly embedded and, hence, has a handle neighborhood M_i of genus g_i such that $M_i \setminus A_i$ is homeomorphic to $S_i \times (0, 1]$, where $S_i = \partial M_i$, and for each point $\sigma \in \mathcal{O}_i$ the intersection $W^s_\sigma \cap M_i$ consists of exactly one 2-disk. Set $D_i = M_i \cap W^s_{\mathcal{O}_i}$. According to statement 6, for each zero-dimensional attractor $A_i,\ i = 1, \dots, k_0$, there is a handle neighborhood of genus $g_i = 0$ which is a union of c_i 3-balls, we will denote it by M_i and set $S_i = \partial M_i$.

For $i = 1, \dots, k_1$ set $K_i = M_i \setminus int\ f(M_i)$, $N_i = W^s_{A_i \cap \Omega_f}$ and $V_i = N_i \setminus A_i$. According to ring hypothesis and corollary 1, K_i is diffeomorphic to $S_i \times [0, 1]$. As $V_i = \bigcup\limits_{n \in \mathbb{Z}} f^n(K_i)$ then V_i is diffeomorphic to $S_i \times \mathbb{R}$.

5.1 Extension of Lyapunov functions

Definition 10 *Let D be a subset of M which is diffeomorphic to product $S \times [0, 1]$ for some (possibly non connected) surface S. Then D is said to be an (f, S)-compressed product when there is a diffeomorphism $g : D \to S \times [0, 1]$ such that $g^{-1}(S \times \{t\})$ bounds an f-compressed domain in M for any $t \in [0, 1]$.*

Proposition 3 *Let D be an (f, S)-compressed product. Then for any values $d_0 < d_1$ there is an energy function $\varphi_{_D} : D \to \mathbb{R}$ for $f|_D$ such that $\varphi_{_D}(g^{-1}(S \times \{0\})) = d_0$ and $\varphi_{_D}(g^{-1}(S \times \{1\})) = d_1$.*

Proof: The desired function $\varphi_{_D} : D \to \mathbb{R}$ is defined by formula $\varphi_{_D}(x) = d_0 + t(d_1 - d_0)$ for $x \in g^{-1}(S \times \{t\}),\ t \in [0, 1]$. $\diamond$

Lemma 1 *Let $i \in \{1, \dots, k_1\}$ and P_i, Q_i be handle neighborhoods of genus g_i of the attractor A_i. If there is a dynamically ordered energy function $\varphi_{_{Q_i}} : Q_i \to \mathbb{R}$ for f with $S_{Q_i} = \partial Q_i$ as a level set then there is a dynamically ordered energy function $\varphi_{_{P_i}} : P_i \to \mathbb{R}$ for f with $S_{P_i} = \partial P_i$ as a level set.*

Proof: We follow to scheme of the proof of lemma 4.2 from [3]. Give some remarks.

Without loss of generality we assume that $Q_i \subset int\ P_i$ (in the opposite case, instead pair (Q_i, φ_{Q_i}) we can use pair $(f^n(Q_i), \varphi_{f^n(Q_i)})$, where $f^n(Q_i) \subset int\ P_i$ and $\varphi_{f^n(Q_i)} = \varphi_{Q_i} f^{-n}$). As V_i is diffeomorphic to $S_i \times \mathbb{R}$ then, according to ring hypothesis and corollary 1, $G_i = P_i \setminus int\, Q_i$ is a product. As handle neighborhoods $f^n(Q_i)$ and $f^n(P_i)$ contain the attractor A_i for each $n \in \mathbb{Z}$ then the surfaces $f^n(S_{Q_i})$ and $f^n(S_{P_i})$ do not bound domains in V_i and, hence, are incompressible due to corollary 1. Now let us construct the function φ_{P_i}, for this aim we consider two cases: 1) $S_{P_i} \cap (\bigcup\limits_{n>0} f^{-n}(S_{Q_i})) = \emptyset$ and 2) $S_{P_i} \cap (\bigcup\limits_{n>0} f^{-n}(S_{Q_i})) \neq \emptyset$.

In case 1), let m be the first positive integer such that $f^m(P_i) \subset int\, Q_i$. If $m = 1$, then G_i is (f, S_i)-compressed product and proposition 3 yields the required function as extension of the function φ_{Q_i} to G_i.

If $m > 1$, the surfaces $f(S_{P_i}), f^2(S_{P_i}), \dots, f^{m-1}(S_{P_i})$ are mutually "parallel", that is: two by two they bound a product cobordism (according to ring hypothesis and corollary 1). Therefore they subdivide G_i in (f, S_i)-compressed products and, hence, proposition 3 yields the required function as extension of the function φ_{Q_i} to $f^{m-1}(P_i) \setminus int\ Q_i, f^{m-2}(P_i) \setminus int\ f^{m-1}(P_i), \dots, P_i \setminus int\ f(P_i)$ in series.

In case 2), without loss of generality we may assume that S_{P_i} is transverse to $\bigcup\limits_{n>0} f^{-n}(S_{Q_i})$, which implies that there is a finite family $\mathcal{C}$ of intersection curves. We are going to describe a process of decreasing the number of intersection curves by an isotopy of Q_i among handle neighborhoods of genus g_i possessing a dynamically ordered energy function for f which is constant on the boundary of the neighborhood.

Firstly we consider all intersection curves from $\mathcal{C}$ which are homotopic to zero in S_{P_i}. Let c be an innermost such curve. Then there is a disc $\delta \subset S_{P_i}$ which is bounded by c and such that $int\, \delta$ contains no curves from the family $\mathcal{C}$. As $c \subset f^{-n}(S_{Q_i})$ for some n and $f^{-n}(S_{Q_i})$ is incompressible in V_i, then c bounds a disc $d \subset f^{-n}(S_{Q_i})$. Then 2-sphere $\delta \cup d$ is embedded and bounds a 3-ball b in N_i (when the component of $f^{-n}(S_{Q_i})$ containing d is a 2-sphere, replace d by the complementary disc if necessary).

There are two occurrences: (a) $f^n(b) \subset Q_i$ and (b) $f^n(b) \subset f^{-1}(Q_i)$. We define Q'_i as $cl(Q_i \setminus f^n(b))$ in the case (a) and $Q_i \cup f^n(b)$ in the case (b). The fact that c is an innermost curve implies $f(Q_i) \subset Q'_i \subset Q_i$ in case (a) and $Q_i \subset Q'_i \subset f^{-1}(Q_i)$ in case (b). In both cases there is a smooth approximation $\tilde{Q}_i$ of Q'_i such that $f(Q_i) \subset int\ \tilde{Q}_i \subset Q_i$ if (a), $Q_i \subset int\ \tilde{Q}_i \subset f^{-1}(Q_i)$ if (b), and the number of intersection curves in $S_{P_i} \cap (\bigcup\limits_{n>0} f^{-n}(\partial \tilde{Q}_i))$ is less than the cardinality of $\mathcal{C}$.

In case (a), $\varphi_{f(Q_i)} = \varphi_{Q_i} f^{-1} : f(Q_i) \to \mathbb{R}$ is a dynamically ordered energy function which is constant on the boundary. Therefore $\tilde{Q}_i \setminus int\ f(Q_i)$ is an (f, S_i)-compressed product and, hence, due to proposition 3, there is a similar function on $\tilde{Q}_i$. Similarly in case (b), $\tilde{Q}_i$ is equipped with a dynamically ordered energy function which is constant on the boundary as $\tilde{Q}_i \setminus int\ Q_i$ is an (f, S_i)-compressed product.

We will repeat this process until getting a handle neighborhood $\hat{Q}_i$ of genus g_i for the attractor A_i such that $S_{P_i} \cap (\bigcup\limits_{n>0} f^{-n}(\partial\hat{Q}_i))$ does not contain curves which are homotopic zero in S_{P_i}. Thus we may assume that $S_{P_i} \cap (\bigcup\limits_{n>0} f^{-n}(S_{Q_i}))$ does not contain intersection curves which are homotopic to zero in S_{P_i}.

We denote by m the largest integer such that $f^m(S_{P_i}) \cap S_{Q_i} \neq \emptyset$. Let F be a connected component of $f^m(S_{P_i}) \cap G_i$. We have $\partial F \subset \partial S_{Q_i}$. Let us show that F is incompressible in G_i. Indeed, if δ is a disc in G_i with boundary $\gamma \subset F$ then γ bounds 2-disk $\tilde{\delta} \subset f^m(S_{P_i})$ as $f^m(S_{P_i})$ is incompressible surface in V_i. By assumption the components of ∂F are not homotopic to zero in $f^m(S_{P_i})$, then $\partial F \cap \tilde{\delta} = \emptyset$ and, hence, $\tilde{\delta} \subset F$.

Therefore, according to statement 2 there is some surface $F_1 \subset S_{Q_i}$ diffeomorphic to F, with $\partial F = \partial F_1$, and $F \cup F_1$ bounds a domain Δ in G_i which, up to smoothing of the boundary, is diffeomorphic to $F \times [0, 1]$. We then define $\tilde{Q}_i$ as $Q_i \cup \Delta$ up to smoothing. By the choice of m, $\tilde{Q}_i$ is f-compessed as $f(\Delta) \subset Q_i$. As $\tilde{Q}_i$ is obtained by an isotopy supported in a neighborhood of Δ from Q_i then $\tilde{Q}_i \setminus int\ Q_i$ is an (f, S_i)-compressed product. Thus we get a dynamically ordered energy function on $\tilde{Q}_i$ with $\partial\tilde{Q}_i$ as a level set. Arguing recursively, we are reduced to case 1). $\diamond$

Lemma 2 *Let $i \in \{k_0 + 1, \dots, k_1\}$, M_i be strongly tight neighborhood of the attractor A_i, $D_i = M_i \cap W^s_{\mathcal{O}_i}$ and $N(D_i) \subset M_i$ be a tubular neighborhood of D_i such that $N(D_i) \cap A_{i-1} = \emptyset$ and the set $P_{i-1} = M_i \setminus int\ N(D_i)$ is f-compressed. Then P_{i-1} is a handle neighborhood of genus g_{i-1} for the attractor A_{i-1}.*

Proof: Similar to proposition 1 it is proved the equality $\bigcap\limits_{k\geq 0} f^k(P_{i-1}) = A_{i-1}$. Thus P_{i-1} is a trapping neighborhood of the attractor A_{i-1} and, hence, the set P_{i-1} consists of c_{i-1} connected components. Each of them is handlebody, as it is obtained from M_i removing $(r_i - r_{i-1})$ 1-handles, which are the set $N(D_i)$. As in proof of proposition 1, a sum $g_{P_{i-1}}$ of genera P_{i-1} is calculated by formula $g_{P_{i-1}} = g_i - ((r_i - r_{i-1}) - (c_{i-1} - c_i))$. Then $g_{P_{i-1}} = c_i + r_i - s_i - ((r_i - r_{i-1}) - (c_{i-1} - c_i)) = c_{i-1} + r_{i-1} - s_i$. As $s_{i-1} = s_i$ then $g_{P_{i-1}} = g_{i-1}$. $\diamond$

5.2 Global construction

We divide a construction of the dynamically ordered energy function for $f : M \to M$ on steps.

Step 1. By induction on $i = 1, \ldots, k_1$ let us prove the existence of a dynamically ordered energy function $\varphi_{_{M_i}}$ on M_i of the attractor A_i with level set S_i.

For $i = 1$ the attractor A_1 coincides with sink orbit $\mathcal{O}_1$ of the diffeomorphism f. According to statement 6 there is a neighborhood $U_{\mathcal{O}_1} \subset M$ of the orbit $\mathcal{O}_1$, equipped by an energy function $\varphi_{\mathcal{O}_1} : U_{\mathcal{O}_1} \to \mathbb{R}$ for f and such that $\varphi_{\mathcal{O}_1}(\mathcal{O}_1) = 1$. Moreover, for each connected component U_ω, $\omega \in \mathcal{O}_1$ of the set $U_{\mathcal{O}_1}$ there are Morse coordinates (x_1, x_2, x_3) such that $\varphi_{\mathcal{O}_1}(x_1, x_2, x_3) = 1 + x_1^2 + x_2^2 + x_3^2$. Then there is a value $\varepsilon_1 > 0$ such that set $Q_1 = \varphi_{\mathcal{O}_1}^{-1}(1 + \varepsilon_1)$ consists of f-compressed union of c_1 3-balls. Thus Q_1 is a handle neighborhood of genus 0 for the attractor A_1. As $g_1 = 0$ then, according to lemma 1, there is a dynamically ordered energy function $\varphi_{_{M_1}}$ on the neighborhood M_1 for the attractor A_1 with level set S_1.

Let, by assumption of the induction, there is a dynamically ordered energy function $\varphi_{_{M_{i-1}}}$ on the neighborhood M_{i-1} of the attractor A_{i-1} with level set S_{i-1}. Let us construct the function $\varphi_{_{M_i}}$. There is two cases: a) $i \le k_0$; b) $i > k_0$.

In the case a) the neighborhood M_i consists of a handle neighborhood of genus 0 for the attractor A_{i-1} (denote it P_{i-1}) and a trapping neighborhood of the orbit $\mathcal{O}_i$, consisting from 3-balls (denote it Q_i). By assumption of the induction and lemma 1 there is a dynamically ordered energy function $\varphi_{_{P_{i-1}}}$ on P_{i-1} with level set ∂P_{i-1}. Similar to case $i = 1$ it is shown the existence of a dynamically ordered energy function $\varphi_{_{Q_i}}$ on Q_i with level set ∂Q_i. The required function $\varphi_{_{M_i}}$ is formed from $\varphi_{_{P_{i-1}}}$ and $\varphi_{_{Q_i}}$.

b) we follow to scheme of proof from section 4.3 of paper [3].

According to statement 6, the orbit $\mathcal{O}_i$ has a neighborhood $U_{\mathcal{O}_i} \subset M$ endowed with an energy function $\varphi_{\mathcal{O}_i} : U_{\mathcal{O}_i} \to \mathbb{R}$ of f with $\varphi_{\mathcal{O}_i}(\mathcal{O}_i) = i$. Moreover, each connected component U_σ, $\sigma \in \mathcal{O}_i$ of $U_{\mathcal{O}_i}$ is endowed with Morse coordinates (x_1, x_2, x_3) such that $\varphi_{\mathcal{O}_i}(x_1, x_2, x_3) = i + x_1^2 + x_2^2 + x_3^2$, the x_1-axis is contained in the unstable manifold and the (x_2, x_3)-plane is contained in the stable manifold of σ.

It follows from properties of strongly tight neighborhood M_i and λ-lemma that there is a tubular neighborhood $N(D_i) \subset M_i$ of $D_i = M_i \cap W^s_{\mathcal{O}_i}$ such that $N(D_i) \cap A_{i-1} = \emptyset$, set $P_{i-1} = M_i \setminus int\ N(D_i)$ is f-compressed and surface ∂P_{i-1} transversal intersects each connected component of the set $\varphi_{\mathcal{O}_i}^{-1}(i) \setminus \mathcal{O}_i$ at one closed curve. By lemma 2, the set P_{i-1} is a handle neighborhood of genus g_{i-1} for the attractor A_{i-1}. By assumption of induction and lemma 1 there is a dynamically ordered energy function $\varphi_{_{P_{i-1}}}$ on P_{i-1} with level set ∂P_{i-1}.

For $\varepsilon_i \in (0, 1)$, $t \in [-\varepsilon_i, \varepsilon_i]$ set $P_t = \varphi_{P_{i-1}}^{-1}([1, \varphi_{_{P_{i-1}}}(\partial P_{i-1}) - \varepsilon_i + t])$, $H_t = \{x \in U_{\mathcal{O}_i} : \varphi_{\mathcal{O}_i}(x) \le i + t\}$ and $E_{\varepsilon_i} = (P_{\varepsilon_i} \setminus int\ P_{-\varepsilon_i}) \cap (H_{\varepsilon_i} \setminus int\ H_{-\varepsilon_i})$ (see figure 5). Notice that $P_{\varepsilon_i} = P_{i-1}$ and, hence, $f(P_{\varepsilon_i}) \subset int\ P_{\varepsilon_i}$. As $\varphi_{\mathcal{O}_i}$ is a Lyapunov function for $f|_{U_{\mathcal{O}_i}}$ then $\varphi_{\mathcal{O}_i}(f^{-1}(\varphi_{\mathcal{O}_i}^{-1}(i) \setminus \mathcal{O}_i)) > i$ and, hence,

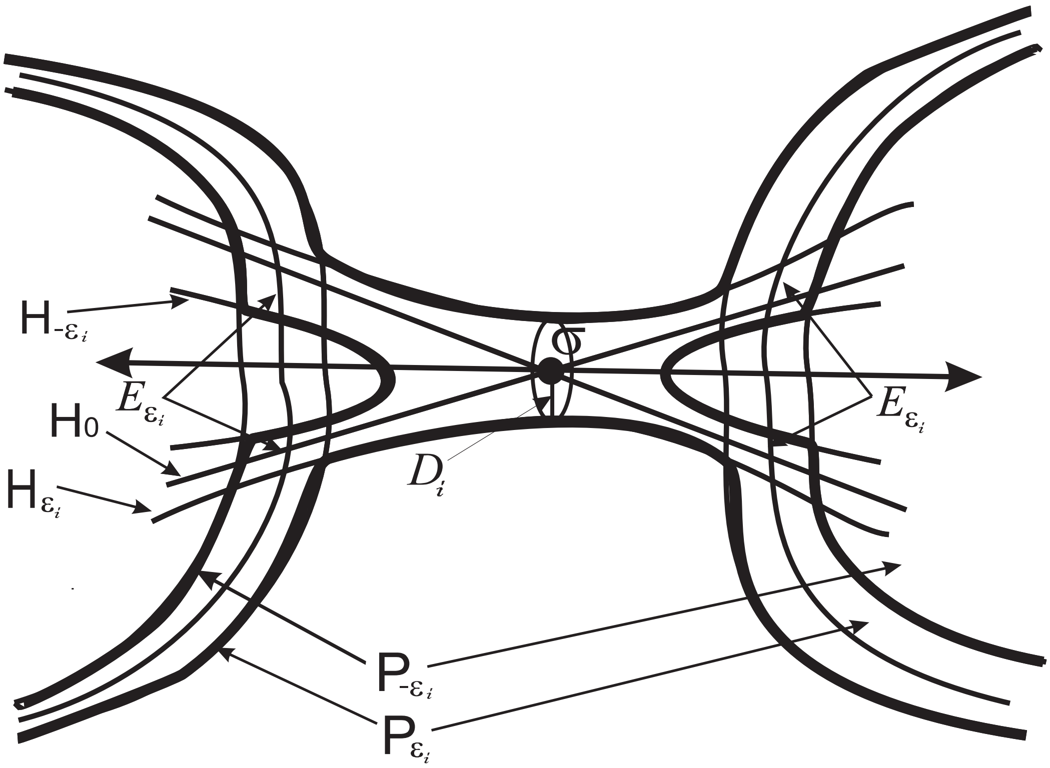

Figure 5: Illustration to step 1

$(H_0 \setminus \mathcal{O}_i) \subset int\ f^{-1}(H_0 \setminus \mathcal{O}_i)$. This and conditions of choice of $N(D_i)$ implies the existence of a value ε_i with following properties:

(1) $f(P_{\varepsilon_i}) \subset int\ P_{-\varepsilon_i}$;

(2) for each $t \in [-\varepsilon_i, \varepsilon_i]$ surface ∂P_t transversal intersects each connected component of the set $\partial H_t \setminus D_i$ at one closed curve;

(3) $f^{-1}(E_{\varepsilon_i}) \cap H_{\varepsilon_i} = \emptyset$.

For $t \in [-\varepsilon_i, \varepsilon_i]$ set $Q_t = P_t \cup H_t$. By the construction the set $Q_t,\ t \neq 0$ is f-compressed. Moreover, $Q_{-\varepsilon_i}$ after smoothing is a handle neighborhood of genus g_{i-1} of the attractor A_{i-1} and Q_{ε_i} after smoothing is strongly tight neighborhood of the attractor A_i. By assumption of induction and lemma 1 there is a dynamically ordered energy function $\varphi_{Q_{-\varepsilon_i}}$ on $Q_{-\varepsilon_i}$, which is a constant on $\partial Q_{-\varepsilon_i}$. As $\varphi_{Q_{-\varepsilon_i}}(A_{i-1}) \leq i-1$ then, due to proposition 3, we can suppose that $\varphi_{Q_{-\varepsilon_i}}(Q_{-\varepsilon_i}) = i - \varepsilon_i$.

Define function $\varphi_{Q_{\varepsilon_i}} : Q_{\varepsilon_i} \to \mathbb{R}$ on the set Q_{ε_i} by formula:
$\varphi_{Q_{\varepsilon_i}}(x) = \begin{cases} \varphi_{Q_{-\varepsilon_i}}(x),\ x \in Q_{-\varepsilon_i}; \\ i+t,\ x \in Q_t. \end{cases}$ Let us check that $\varphi_{Q_{\varepsilon_i}}$ is a dynamically ordered energy function f, then the existence of required function $\varphi_{M_i} : M_i \to \mathbb{R}$ will follow from lemma 1.

Represent the set Q_{ε_i} as a union of subsets with pairwise disjoint interiors: $Q_{\varepsilon_i} = A \cup B \cup C$, where $A = Q_{-\varepsilon_i}$, $B = P_{\varepsilon_i} \setminus Q_{-\varepsilon_i}$ and $C = Q_{\varepsilon_i} \setminus (P_{\varepsilon_i} \cup Q_{-\varepsilon_i})$. By the construction $\varphi_{Q_{\varepsilon_i}}|_A$ a dynamically ordered energy function for f, $\varphi_{Q_{\varepsilon_i}}(\partial A) = i - \varepsilon_i$, the function $\varphi_{Q_{\varepsilon_i}}|_B$ has no critical points and function $\varphi_{Q_{\varepsilon_i}}|_C$ coincides with function $\varphi_{\mathcal{O}_i}|_C$. Let us check decreasing property of $\varphi_{Q_{\varepsilon_i}}$ along trajectories of f.

If $x \in A$ then $f(x) \in A$ and $\varphi_{Q_{\varepsilon_i}}(f(x)) < \varphi_{Q_{\varepsilon_i}}(x)$, as $\varphi_{Q_{\varepsilon_i}}|_A$ is a Lyapunov function. If $x \in B$ then, due to condition (1) of choice of ε_i, $f(x) \in A$ and, hence, $\varphi_{Q_{\varepsilon_i}}(x) > i - \varepsilon_i$, $\varphi_{Q_{\varepsilon_i}}(f(x)) < i - \varepsilon_i$, therefor $\varphi_{Q_{\varepsilon_i}}(f(x)) < \varphi_{Q_{\varepsilon_i}}(x)$. If $x \in C$ then, due to condition (3) of choice of ε_i, either $f(x) \in A$ and decreasing is proved as for $x \in B$, or $f(x) \in C$ and decreasing follows from the fact that $\varphi_{Q_{\varepsilon_i}}|_C$ is a Lyapunov function.

Step 2. In this step we delive a construction similar to step 1 for diffeomorphism f^{-1}. For this aim we recall that dynamical numbering of the orbits $\mathcal{O}_1, \dots, \mathcal{O}_{k_f}$ of the diffeomorphism f induces dynamical numbering of the orbits $\tilde{\mathcal{O}}_1, \dots, \tilde{\mathcal{O}}_{k_f}$ of the diffeomorphism f^{-1} following way: $\tilde{\mathcal{O}}_i = \mathcal{O}_{k_f - i}$. Denote by $\tilde{A}_i$ the attractors of the diffeomorphism f^{-1}, by $\tilde{M}_i$ thir neighborhood and by $\tilde{g}_i$ a number, defined by formula $\tilde{g}_i = \tilde{c}_i + \tilde{r}_i - \tilde{s}_i$, wher $\tilde{c}_i$ the number of the connected components of the attractor $\tilde{A}_i$, $\tilde{r}_i$ the number of the saddle pointd and $\tilde{s}_i$ the number of the sink points of the diffeomorphism f^{-1}, belonging to $\tilde{A}_i$.

Set $\tilde{k}_1 = k_f - k_1$ and consider the attrator $\tilde{A}_{\tilde{k}_1}$ for the diffeomorphism f^{-1} (which, recall, is a repeller for the diffeomorphism f). Similar to step 1 we construct a a dynamically ordered energy function $\tilde{\varphi}_{\tilde{M}_{\tilde{k}_1}}$ for f^{-1} on the neighborhood $\tilde{M}_{\tilde{k}_1}$ with level set $\tilde{S}_{\tilde{k}_1} = \partial \tilde{M}_{\tilde{k}_1}$.

Step 3. In this step we show that set $P_{k_1} = M \setminus int\ \tilde{M}_{\tilde{k}_1}$ is a handle neighborhood of genus g_{k_1} of the attractor A_{k_1}, this implies the existence of the required function φ. Indeed, by lemma 1, the existence of a dynamically ordered energy function $\varphi_{M_{k_1}}$ on the neighborhood M_{k_1} of the attractor A_{k_1} implies the existence of a dynamically ordered energy function for $\varphi_{P_{k_1}}$ on P_{k_1} with level set ∂P_{k_1}. According to proposition 3 the function $\varphi_{P_{k_1}}$ we can construct such that $\varphi_{P_{k_1}}(\tilde{S}_{\tilde{k}_1}) = k_f + 1 - \tilde{\varphi}_{\tilde{M}_{\tilde{k}_1}}(\tilde{S}_{\tilde{k}_1})$. As $\partial P_{k_1} = \tilde{S}_{\tilde{k}_1}$ then required function φ is defined by formula $\varphi(x) = \begin{cases} \varphi_{P_{k_1}}(x), \ x \in P_{k_1}; \\ k_f + 1 - \tilde{\varphi}_{\tilde{M}_{\tilde{k}_1}}(x), \ x \in \tilde{M}_{\tilde{k}_1}. \end{cases}$

Thus, let us prove that the set $P_{k_1} = M \setminus int\ \tilde{M}_{\tilde{k}_1}$ is a handle neighborhood of genus g_{k_1} of the attractor A_{k_1}. Set $\tilde{N}_{\tilde{k}_1} = W^s_{\tilde{A}_{\tilde{k}_1} \cap \Omega_{f^{-1}}}$ and $\tilde{V}_{\tilde{k}_1} = \tilde{N}_{\tilde{k}_1} \setminus \tilde{A}_{\tilde{k}_1}$. Notice that the open sets V_{k_1} and $\tilde{V}_{\tilde{k}_1}$ are coincide, as both are obtained from M by removing of A_{k_1} and $\tilde{A}_{\tilde{k}_1}$. It follows from proof of proposition 2 that each of next sets $A_{k_1},\ \tilde{A}_{\tilde{k}_1}, M_{k_1},\ \tilde{M}_{\tilde{k}_1},\ N_{k_1},\ \tilde{N}_{\tilde{k}_1},\ V_{k_1},\ \tilde{V}_{\tilde{k}_1}$ is connected. Then

$g_{k_1} = 1 + |\Omega_1| - |\Omega_0|$ and $\tilde{g}_{\tilde{k}_1} = 1 + |\Omega_2| - |\Omega_3|$. From statement 3 we get $g_{k_1} = \tilde{g}_{\tilde{k}_1}$. Thus the handle neighborhoods M_{k_1} and $\tilde{M}_{\tilde{k}_1}$ have the same genera and their boundaries S_{k_1} and $\tilde{S}_{\tilde{k}_1}$ belong to the set V_{k_1}, which is diffeomorphic to $S_{k_1} \times \mathbb{R}$.

Choose $n \in \mathbb{N}$ such that $f^n(M_{k_1}) \subset int\ P_{k_1}$. Then, according to ring hypothesis and corollary 1, manifold $K = P_{k_1} \setminus int\ f^n(M_{k_1})$ is diffeomorphic to $S_{k_1} \times [0,1]$. By the construction $f^n(M_{k_1})$ is a handle neighborhood of genus g_{k_1} of the attractor A_{k_1} and $P_{k_1} = f^n(M_{k_1}) \cup K$. This implies that P_{k_1} also is handle neighborhood of genus g_{k_1} of the attractor A_{k_1}.

6 Dynamically ordered energy function for diffeomorphisms on 3-sphere

In this section $f : \mathbb{S}^3 \to \mathbb{S}^3$ is a Morse-Smale diffeomorphism without heteroclinic curve.

Proof of theorem 3

Let us prove that diffeomorphism f possesses a dynamically ordered energy function if and only if all its one-dimensional attractors and repellers are tightly embedded.

Proof: The necessity of conditions of the theorem follows from 1, let us proof the sufficiency.

Let $i = k_0 + 1, \dots, k_1$. Then A_i is one-dimensional attractor, consisting of c_i connected components, containing r_i saddles, s_i sinks and for which a number g_i can be calculated by formula $g_i = c_i + r_i - s_i$. Firstly prove that $g_i = 0$ for each $i = k_0 + 1, \dots, k_1$.

We start from g_{k_1}. According to proposition 2, the attractor A_{k_1} is connected that is $m_{k_1} = 1$ and, hence, $g_{k_1} = 1 + |\Omega_1| - |\Omega_0|$. Due to statement 3, we have $g_{k_1} = \tilde{g}_{k_1}$, where $\tilde{g}_{k_1} = 1 + |\Omega_2| - |\Omega_3|$. According to statement 7, $2 + |\Omega_1 \cup \Omega_2| - |\Omega_0 \cup \Omega_3| = 0$ for any Morse-Smale diffeomorphism without heteroclinic curves on $\mathbb{S}^3$. Thus $g_{k_1} + \tilde{g}_{k_1} = 0$ and, hence, $g_{k_1} = \tilde{g}_{k_1} = 0$. Further let us show that $g_i \leq g_{i+1}$ for each $i = k_0, \dots, k_1 - 1$.

Indeed, $g_{i+1} - g_i = (c_{i+1} - c_i) + (r_{i+1} - r_i) - (s_{i+1} - s_i)$. At the same time $(c_i - c_{i+1}) \leq (r_{i+1} - r_i)$, $s_{i+1} = s_i$ and, hence, $g_{i+1} \geq g_i$.

Thus, $g_i = 0$ for each $i = k_0 + 1, \dots, k_1$. Then $K_i = M_i \setminus int\ f(M_i)$ is a union of 3-dimensional annulus $S^2 \times [0,1]$. As $M_i \setminus A_i = \bigcup\limits_{k \geq 0} f^k(M_i)$ then $M_i \setminus A_i$ is diffeomorphic to $\partial M_i \times (0,1]$. Thus, the attractor A_i is strongly tight embedded. Similar fact has place for repellers. The, according to theorem 2, f possesses a dynamically ordered energy function. $\diamond$

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